HYPERREAL WAVES ON TRANFINITE, TERMINATED, DISTORTIONLESS AND LOSSLESS, TRANSMISSION LINES

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Abstract — A prior work examined the propagation of an electromagnetic wave on a transfinite transmission line—transfinite in the sense that infinitely many one-way infinite transmission lines are connected in cascade. That there are infinitely many such lines results in the wave propagating without ever reflecting at some discontinuity This work examines the case where the line is terminated after finitely many one-way infinite transmission lines with the result that reflected waves are now produced at both the far end as well as at the initial end of the transmisssion line. The questions of whether the reflected waves are infinitesimal or appreciable and whether they sum to an infinitesimal or appreciable amount are resolved for both distortionless and lossless lines. Finally, the generalization to higher ranks of transfiniteness is briefly summarized

Key Words: transfinite transmission lines, hyperreal waves, nonstandard transients, nonstandard analysis of transmission lines.

1 Introduction

Consider a one-way-infinite electromagnetic transmission line. The voltage $v(x,t)$ and current $i(x,t)$ at the point x and at time t are governed by the usual wave equations

$$\frac{\partial v(x,t)}{\partial x} = -\left(r + l\frac{\partial}{\partial t}\right) i(x,t)$$

$$\frac{\partial i(x,t)}{\partial x} = -\left(g + c\frac{\partial}{\partial t}\right) v(x,t)$$

along with some initial and boundary conditions. r, l, g, and c are respectively the distributed series resistance, series inductance, shunt conductance, and shunt capacitance per unit length of the line. Such a conventional, one-way-infinite, transmission line will be called an *ω-line*.

A natural transfinite extension of an ω-line is an infinite cascade of ω-lines, wherein the infinite extremity of each ω-line is connected to the input of the next ω-line in the cascade, as is indicated in Figure 1. We call such a structure an *ω^2-line*. Of course, it takes an infinitely long time for any wave to pass completely through the initial ω-line. Furthermore, a wave will in general (but need not) decay to an infinitesimal size when it passes beyond the initial ω-line. In any case, the wave on subsequent ω-lines can be analyzed by means of nonstandard analysis. In particular, real time t can be extended into unlimited hyperreal time $\mathbf{t}$, real distance along the initial ω-line can be extended into unlimited hyperreal distance $\mathbf{x}$ along the ω^2-line, and the voltage $\mathbf{v}(\mathbf{x}, \mathbf{t})$ of the wave along the ω^2-line is given as an "internal function" $\mathbf{v}$ of $\mathbf{x}$ and $\mathbf{t}$.

An important property of the ω^2-line is that it, too, is one-way infinite in the sense that it is a cascade of infinitely many ω-lines. Thus, the wave front of the voltage wave $\mathbf{v}(\mathbf{x}, \mathbf{t})$ never reaches the infinite exgtremity of the $\mathbf{v}(\mathbf{x}, \mathbf{t})$. No matter how large $\mathbf{t}$ is, that wave front will have propagated only part of the way along the ω^2-line. As a result, there is no reflected wave returning from that extremity. This is the situation examined in the prior work [2].

On the other hand, we might think of a cascade of only finitely many ω-lines with some terminating load resistance at the far end of that cascade. We will call such a structure a *terminated transfinite line*. It will have reflected waves. In fact, reflections may occur at both the sending end ($\mathbf{x} = 0$) and the receiving end (the far end of the line) so that more and more waves fronts may pass forward and backward past any fixed point $\mathbf{x}$ of the line as $\mathbf{t}$ increases. In general, those waves will be infinitesimally small, but not necessarily; they may appreciable. Moreover, for time $\mathbf{t}$ being a sufficiently large unlimited hyperreal, all the reflected waves will be present at a point $\mathbf{x}$, and their sum may or may not be infinitesimal. Furthermore, these ideas also hold for terminated transmission lines having higher ranks of

transfiniteness.

The objective of this paper is to analyze such terminated transfinite lines.

This paper is written as a sequel to [2]. We will borrow some results from that prior paper but will explain what we are borrowing. Furthermore, we will use some notations, terminology, and results of nonstandard analysis; these were specified in the Introduction of [2] and will not be repeated here.

2 Truncations of ω^2-Lines and an Apparent Anomaly

Consider the ω^2-line of Figure 1. We wish to approximate it by an ω-line in order to use the known results concerning waves on ω-lines. We do so as in [2, Section 4] by choosing a set of uniformly spaced sample points a distance Δx apart and indexed by $j = \omega k_1 + k_0$. Here, ω is the first transfinite ordinal. Also, $k_0, k_1 \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers: $\{0, 1, 2, \ldots\}$ For the jth sample point in the ω^2-line, k_1 is the number of ω-lines to the left of the ω-line in which that jth sample point appears, and k_0 is the number of sample points to the left of the jth sample point in the ω-line in which that jth sample point appears. To obtain the nth ω-line approximation of the given ω^2-line, we remove the infinite part of each ω-line in the ω^2-line beyond the nth sample point of that ω-line and reconnect the resulting finite lines in cascade. We call this resulting ω-line the nth *truncation* of the ω^2-line. Thus, as $n \to \infty$, each sample point eventually appears in the nth truncation. When this occurs, the number of sample points to the left of the jth sample point is $nk_1 + k_0$.

With Δx being the distance between consecutive sample points, the distance $x_{j,n}$ from the input to the jth sample point in the nth truncation is

$$x_{j,n} = (nk_1 + k_0)\Delta x \in \mathbb{R}_+,$$

where $\mathbb{R}_+$ denotes the nonnegative real line.

Let us henceforth assume that a voltage has been imposed upon the input of the ω^2-line during time $t \geq 0$, with that line being initially at rest at $t = 0$. This induces a wave that propagates along the ω^2-line. With regard to the nth truncation, we have a wave that propagates along the nth approximating ω-line, about which much analysis exists;

see for instance [1]. When n is large enough for the jth sample point to appear in the nth truncation, we have the voltage response at the jth sample point as $v(x_{j,n}, t)$. For any positive hyperreal time $\mathbf{t} = [t_n] > 0$ and hyperreal distance $\mathbf{x}_j = [x_{j,n}] = [(nk_1 + k_0)\Delta x] \geq 0$, we obtain the hyperreal voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ at the point $\mathbf{x}_j$ in the ω^2-line at the time $\mathbf{t}$ specified by the representative sequence $\langle v(x_{j,n}, t) \rangle$ indexed by n for all n sufficiently large. In the usual symbolism for hyperreals, we have

$$\mathbf{v}(\mathbf{x}_j, \mathbf{t}) \;=\; [v(x_{j,n}, t)].$$

Note that there will be no reflected wave appearing at $\mathbf{x}_j$ because no matter how large we choose $\mathbf{t} = [t_n]$ there will be more of the ω^2-line further on beyond the wave front of $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$. Indeed, for each n, the wave will have propagated only a finite distance along the nth truncation.

However, there is an apparent—but not actual—anomaly that seems to arise now. Consider two sample points $\mathbf{x}_i = [x_{i,n}]$ and $\mathbf{x}_j = [x_{j,n}]$, where $x_{i,n} = (nh_1 + h_0)\Delta x$ and $x_{j,n} = (nk_1 + k_0)\Delta x$. Moreover, consider the case where $h_1 < k_1$ but h_0 is so much larger than k_0 that $mh_1 + h_0 > mk_1 + k_0$ for some positive but sufficiently small $m \in \mathbb{N}$. Then, for $n \leq m$, $x_{j,n}$ appears in the nth truncation, while $x_{i,n}$ does not. Consequently, the wave front reaches the further (from the input) sample point $x_{j,n}$ in the nth truncation (for $n \leq m$) before it appears at the nearer (from the input) sample point $x_{i,n}$ because the nearer one does not exist in that nth truncation. This seems paradoxical, but it is not. This effect occurs only for finitely many values of n. For all sufficiently large n, both points appear with $x_{i,n} < x_{j,n}$. Consequently, the hyperreal wave front truly reaches $\mathbf{x}_i$ before it reaches $\mathbf{x}_j$.

3 A Terminated Transfinite Line and Its Truncations

The ω^2-line was analyzed in [2], and it had no reflected waves—as was explained above. Our objective in this work is to analyze a transfinite line consisting only of finitely many ω-lines in cascade with perhaps a finite line as the last cascaded line. As a result, a hyperreal wave along such a line will reach its far end at some unlimited hyperreal time $\mathbf{T}$ and undergo a

reflection in general. The reflected wave will then reach the input after another time $\mathbf{T}$ has elapsed, and another reflection might then occur, followed by another propagation along the line, and so on.

More specifically, the model we shall now analyze is shown in Figure 2. It is a transfinite line consisting of l_1 ω-lines in cascade with possibly a final cascaded finite line. We index the 1-nodes between these lines by ω, $\omega 2$, $\ldots$, ωl_1. With k_1 being the number of ω-lines to the left of the ω-line in which the jth sample point appears, we must have $0 \leq k_1 \leq l_1$. Also, with k_0 being the number of sample points to the left of the jth sample point within the ω-line in which the the jth sample point appears, we have $0 \leq k_0 < \infty$. If moreover, the jth sample point appears in the last finite line, we have $0 \leq k_0 \leq l_0$.

The input to the line will be called the *sending end*, and the far end of the line will be called the *receiving end*. Also, the *forward* (resp. *backward*) directions are from the sending (resp. receiving) end toward the receiving (resp. sending) end. Furthermore, we take it that the input is driven by a voltage source of value $w(t)$ in series with a sending-end resistor R_s and a switch which is thrown closed at $t = 0$. So, we restrict time to nonnegative values. Moreover, we assume that the line is initially at rest at $t = 0$ and that there is a resistor R_r terminating the line at its receiving end. We require that $0 \leq R_s < \infty$ and $0 \leq R_r \leq \infty$, with 0 being a possible value for R_s and with 0 and ∞ being possible values for R_r. We also assume that $0 < w(t) \leq M$ for $t \geq 0$, where M is a positive real number (but this condition on $w(t)$ can be relaxed with some adjustments to our conclusions).

The closing of the switch will initiate a wave that propagates forward along the line, reflects at the receiving end, propagates backward, reflects at the receiving end, and so on. It will take an infinite amount of time for the wave front to propagate completely along any ω-line, and the forward and backward reflected waves might be infinitesimally small. It is nonstandard analysis that enables a mathematical examination of all this.

To this end, we consider a sequence of truncations of each ω-line into a finite line of length $n\Delta x$, as was done in [2]. Let us choose Δx such that the length of the last finite line in the original terminated ω^2-line (if such exists) is a natural-number multiple l_0 of Δx. So, for the nth truncation, the total length L_n of the nth truncation of the terminated

line is $L_n = (nl_1 + l_0)\Delta x$. The jth sample point will appear in the nth truncation of the terminated ω^2-line when $n \geq k_0$ because each truncated ω-line is of length $n\Delta x$, while $k_0 \Delta x$ is the distance of the jth sample point from the beginning of the ω-line in which that sample point appears. Henceforth, we take it that $n \geq k_0$, and thus the distance $x_{j,n}$ of the jth sample point from the sending end is

$$x_{j,n} \; = \; (nk_1 \, + \, k_0)\Delta x \; = \; K_n, \tag{1}$$

and its distance $L_n - x_{j,n}$ to the receiving end is

$$L_n \, - \, x_{j.n} \; = \; (n(l_1 \, - \, k_1) \, + \, l_0 \, - \, k_0)\Delta x. \tag{2}$$

Altogether then, each nth truncation is a finite transmission line whose voltage response $v_j(x_{j,n}, t)$ at the jth sample point at any time t is uniquely determined. So, for any hyperreal time $\mathbf{t} = [t_n]$ and with the truncated line expanding as $n \to \infty$ to fill out the ω^2-line, we have a sequence $\langle v_j(x_{j,n}, t_n \rangle$ for all n sufficiently large, which determines a hyperreal response

$$\mathbf{v}(\mathbf{x}_j, \mathbf{t}) \; = \; [v(x_{j,n}, t_n)] \tag{3}$$

at the jth sample point of the ω^2-line. We shall now examine that response more explicitly in terms of the reflected, forward and backward waves. The latter may or may not be infinitesimal depending upon the parameters of the line and the terminating resistors R_s and R_r.

4 A Finite Line

Let us recall the voltage response of a finite transmission line, the one shown in Figure 3. In general, the propagation constant γ is given by

$$\gamma \; = \sqrt{(ls + r)(gs + c)} \; = \; \sqrt{lc}\sqrt{(s + \delta)^2 - \sigma^2}, \quad Re\, s \, > \, 0, \tag{4}$$

where s is the complex variable of the Laplace transform (except when it appears as a subscript) and where

$$\delta \; = \; \frac{1}{2}\left(\frac{r}{l} + \frac{g}{c}\right), \quad \sigma \; = \; \frac{1}{2}\left(\frac{r}{l} - \frac{g}{c}\right). \tag{5}$$

Also, the characteristic impedance of the line is

$$Z_0 \;=\; \sqrt{\frac{ls + r}{cs + g}}. \tag{6}$$

Moreover, with r_s and r_r being the sending-end and receiving-end reflection coefficients respectively, we have

$$r_s \;=\; \frac{R_s - Z_0}{R_s + Z_0}, \qquad r_r \;=\; \frac{R_r - Z_0}{R_r + Z_0}. \tag{7}$$

We are allowing R_s to be a short (i.e., $R_s = 0$, $r_s = -1$) and R_r to be either a short or an open (i.e., $R_r = 0$, $r_r = -1$ or $R_r = \infty$, $r_r = 1$). With this notation and with $W(s)$ being the Laplace of $w(t)$, the Laplace transform $V(x,s)$ of the voltage response $v(x,t)$ at the point x $(0 \leq x \leq L)$ of the line at time $t \geq 0$ is the following:

$$\begin{aligned} V(x,s) \;=\; \frac{Z_0}{Z_0 + R_s} W(s) \quad & (\;\; e^{-\gamma x} \,+\, r_r e^{-\gamma(2L-x)} \\ + \;\; & r_s r_r e^{-\gamma(2L+x)} \,+\, r_s r_r^2 e^{-\gamma(4L-x)} \\ + \;\; & \cdots \\ + \;\; & r_s^m r_r^m e^{-\gamma(2mL+x)} \,+\, r_s^m r_r^{m+1} e^{-\gamma(2(m+1)L-x)} \\ + \;\; & \cdots) \, , \qquad m = 0, 1, 2, \ldots \, . \end{aligned} \tag{8}$$

If the line's parameters γ, R_s, and R_r are such that

$$|r_s \, r_r \, e^{-2\gamma L}| \;<\; 1, \tag{9}$$

this series can be summed to get

$$V(x,s) \;=\; \frac{Z_0}{Z_0 + R_s} W(s) \frac{e^{-\gamma x} \,+\, r_r e^{-2\gamma L} e^{\gamma x}}{1 \,-\, r_s r_r e^{-2\gamma L}}. \tag{10}$$

(An equation much like that of (8) holds for the current on the line; the analysis for it is similar to that for the line voltage.)

An important special case, whose transfinite version we shall examine in some detail, is the distortionless line. This occurs when r, g, l, and c are all positive and such that $\sigma = 0$ and $\delta = r/l = g/c$. Thus, Z_0 is now the real number $\sqrt{l/c}$ and $\gamma = \sqrt{lc}\,(s + \delta)$. Moreover, $-1 \leq r_s < 1$ and $-1 \leq r_r \leq 1$. In this case, the condition (9) is satisfied for $\mathrm{Re}(s + \delta) > 0$, and the Laplace transform (10) is valid. Moreover, we may take the inverse

Laplace transform term by term in (8). Upon setting $\alpha = \sqrt{lc}\,\delta$ and $u = 1/\sqrt{lc}$, we obtain the inverse Laplace transform $v(x,t)$ of $V(x,s)$ as

$$\begin{aligned} v(x,t) \;&=\; \frac{Z_0}{Z_0 + R_s}(e^{-\alpha x} w(t - x/u) \,+\, r_r\, e^{-\alpha(2L-x)}\, w(t - (2L - x)/u) \\ &+\; r_s\, r_r\, e^{-\alpha(2L+x)}\, w(t - (2L + x)/u) \,+\, r_s\, r_r^2\, e^{-\alpha(4L-x)}\, w(t - (4L - x)/u) \\ &+\; \cdots \\ &+\; r_s^m\, r_r^m\, e^{-\alpha(2mL+x)}\, w(t - (2mL + x)/u) \,+\, r_s^m\, r_r^{m+1}\, e^{-\alpha(2(m+1)L-x)}\, w(t - (2(m+1)L - x)/u) \\ &+\; \cdots). \end{aligned} \tag{11}$$

Here, $u = 1/\sqrt{lc}$ is the speed of propagation of the waves.

Another special case occurs when $r = g = 0$, $l > 0$, and $c > 0$. Now, $\alpha = \delta = \sigma = 0$. Equation (11) holds again except that all the exponential damping terms are replaced by 1.

5 A Terminated Transfinite Distortionless Line

We wish to examine the forward and backward voltage waves and also their sum (i.e., the total voltage) at any point of the terminated transfinite line of Figure 2, which we now take to be distortionless. To ensure transfiniteness, we assume that $k_1 \geq 1$, that is, the initial line in the cascade of Figure 2 is truly one-way infinite. In that initial ω-line, the wave front propagates at the speed $u = 1/\sqrt{lc}$ and reaches a point x in the ω-line at the time $t = x/u$. The voltage response is 0 for $t < x/u$ and possibly nonzero for $t > x/u$; in fact; it is positive for $t > x/u$ if $w(t)$ is positive for $t > 0$. In order to examine the propagation of the wave beyond the initial ω-line and its subsequent reflections, we have to extend real time t to unlimited hyperreal time $\mathbf{t} = [t_n]$, where $t_n \to \infty$ as $n \to \infty$. So, we should also extend the real source voltage w to an internal function $\mathbf{w}$ of $\mathbf{t}$ that is identical to w for all real t. Specifically, we set $\mathbf{w}(\mathbf{t}) = [w(t_n)]$. We shall assume that $w(t)$ is bounded for all t, that is, there is an $M \in \mathbb{R}_+$ such that $|w(t)| \leq m$ for all t. Consequently, $\mathbf{w}(\mathbf{t}) \leq M$ for all $\mathbf{t}$ as well.

To ascertain whether the forward and backward waves and their sum are infinitesimal or instead appreciable, we examine the nth truncation of the line and then determine

what occurs when $n \to \infty$. For a more concise notation, we use $L_n = (l_1 n + l_0)\Delta x$ and $x_{j,n} = (k_1 n + k_0)\Delta x$. Thus, L_n is the length of the nth truncated line, and $x_{j,n}$ is the distance of the jth sample point from the sending end for all n large enough to encompass the jth sample point. We always take it that n is sufficiently large this way. In this case, $L_n - x_{j,n}$ is the distance of the jth sample point from the receiving end.

Thus, in accordance with (11), in the nth truncation of the ω^2-line, the voltage at jth sample point at time t_n is obtained from (11) by replacing x by $x_{j,n}$, L by L_n, and t by t_n. According to the first term in the resulting series, the wave front first reaches the jth sample point $\mathbf{x}_j = [x_{j,n}]$ in the transfinite line at the hyperreal time $\mathbf{t} = \mathbf{x}_j/u = [x_{j,n}/u]$. Similarly, after m reflections at both the receiving end and the sending end, the wave front of the mth forward reflected wave reaches that point at time

$$\mathbf{t} \;=\; [(2mL_n \,+\, x_{j,n}/u])$$

Also, after one more reflection at the receiving end, the wave front of the $(m+1)$st backward wave reaches that point at time

$$\mathbf{t} \;=\; [(2(m+1)L_n \,-\, x_{j,n}/u].$$

It follows that, for $\mathbf{t} = [t_n]$ with $t_n = O(n)$, only finitely many reflected waves will have reached any hyperreal sample point $\mathbf{x}_j$. On the other hand, if $t_n/n \to \infty$ as $n \to \infty$, then all of the forward and backward waves will have reached every hyperreal sample point $\mathbf{x}_j$.

So, now consider the hyperreal waves on the terminated transfinite line. The initial forward wave in the initial ω-line remains appreciable and is given by

$$\frac{Z_0}{Z_0 + R_s}[e^{-\alpha x_{j,n}}\, w(t_n - x_{j,n}/u)]$$

at the jth sample point therein, where $x_{j,n} = k_0 \Delta x$. However, that initial forward wave becomes infinitesimal beyond that initial ω-line because now $x_{j,n} = (k_1 n + k + 0)\Delta x$ with $k_1 > 0$, and thus it is no larger in absolute value than $M\, e^{-\alpha(k_1 n + k_0)\Delta x}$, where $\alpha = \sqrt{lc}\, r/l = \sqrt{lc}\, g/c$. Similarly, all the reflected waves are infinitesimal too. Indeed, since $0 \geq x_{j,n} \geq L_n$, the absolute values of the mth reflected forward wave and the $(m+1)$st

reflected backward wave are both bounded by the hyperreal

$$M\,[e^{-\alpha 2mL_n}],$$

where $m \geq 1$, $L_n > 0$, and $L_n \to \infty$ as $n \to \infty$. Thus, all the waves (i.e., the appreciable one in the initial ω-line and all the other infinitesimal ones) diminish at least exponentially as they propagate.

What about the sum of the reflected, forward and backward, waves? In general, an infinite sum of infinitesimals need not be infinitesimal, but in this case it is. To see this, consider any subsequent finite line in the nth truncation, the ones for which $k_1 > 0$. (The argument is much the same for the reflected waves in the initial line.) Remember that $L_n = (l_1 n + l_0)\Delta x$, $x_{j,n} = (k_1 n + k_0)\Delta x$, $l_1 > 0$, and $l_1 n + l_0 - k_1 n - k_0 \geq 0$. Also, $l_0 \geq 0$. So,

$$2(l_1 n + l_0) - (k_1 n + k_0) \;\geq\; l_1 n + l_0 \;\geq l_1 n.$$

Since $|Z_0/(Z_0 + R_s)| \leq 1$, $|r_s| \leq 1$, and $|r_r| \leq 1$, we have the following estimate for the voltage $v(x_{j,n}, t_n)$ of the nth truncation at the jth sample point $x_{j,n}$ beyond the initial truncated ω-line:

$$\begin{aligned}
|v(x_{j,n}, t_n)| \;\leq\; M \quad \Big(\; & e^{-\alpha(k_1 n + k_0)\Delta x} + e^{-\alpha(2(l_1 n + l_0) - (k_1 n + k_0))\Delta x} \\
+ \; & e^{-\alpha(2(l_1 n + l_0) + (k_1 n + k_0))\Delta x} + e^{-\alpha(4(l_1 n + l_0) - (k_1 n + k_0))\Delta x} \\
+ \; & \cdots \\
+ \; & e^{-\alpha(2m(l_1 n + l_0) + (k_1 n + k_0))\Delta x} + e^{-\alpha(2(m+1)(l_1 n + l_0) - (k_1 n + k_0))\Delta x} \\
+ \; & \cdots \Big)
\end{aligned}$$

$$\begin{aligned}
\leq\; M \quad \Big(\; & e^{-\alpha k_1 n \Delta x} + e^{-\alpha l_1 n \Delta x} \\
+ \; & e^{-\alpha(2l_1 + k_1)n\Delta x} + e^{-\alpha 3 l_1 n \Delta x} \\
+ \; & \cdots \\
+ \; & e^{-\alpha(2ml_1 + k_1)n\Delta x} + e^{-\alpha(2m+1)l_1 n \Delta x} \\
+ \; & \cdots \; \Big)
\end{aligned}$$

$$= M \left(\frac{e^{-k_1 n\Delta x}}{1 - e^{-\alpha 2 l_1 \Delta x}} + \frac{e^{-l_1 n\Delta x}}{1 - e^{-\alpha 2 l_1 \Delta x}} \right).$$

So, with regard to the ω^2-line now, since the right-hand side tends to 0 as $n \to \infty$, it follows that, at any sample point beyond the first ω-line, not only each forward or backward wave is infinitesimal but their sum is infinitesimal, too.

This argument also shows that the reflected, forward and backward, waves at any sample point in the initial ω-line also sum to an infinitesimal.

Since there is attenuation of the wave in a distortionless line with $r > 0$ and $g > 0$, these conclusions are unsurprising at least for the individual, reflected, forward and backward, waves. Things are different for a lossless line.

6 A Terminated Transfinite Lossless Line

We now consider the lossless case of an ω^2-line. This occurs when $r = g = 0$, $l > 0$, and $c > 0$ so that $\alpha = 0$, too. Now, there is no attenuation of the waves, but otherwise the waves propagate as they do in the distortionless case. All of the forward and backward waves take on appreciable values at the sample points; they do not assume nonzero infinitesimal values. As before, when $t_n = O(n)$, $\mathbf{t} = [t_n]$, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is at most a finite sum of appreciable values. If however $t_n/n \to \infty$ as $n \to \infty$, $\mathbf{v}(\mathbf{x}_j, \mathbf{t}) = [v(x_{j,n}, t_n)]$ will in general be determined by an infinite sequence of real, nonzero values $v(x_{j,n}, t_n)$. Just which hyperreal should be assigned to $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ will depend upon which nonprincipal ultrafilter is chosen in our ultrapower construction of hyperreals.

To illustrate these ideas, let us consider two examples. As always, we assume that $k_1 \geq 1$.

Example 6.1. For the sending-end source $w(t)$, we now choose the unit-step function $1_+(t) = 0$ for $t < 0$ and $1_+(t)$ for $t \geq 0$. Thus, each wave propagates as a unit step. Let $0 < R_s < \infty$ and $R_r = \infty$. That is, we have a positive resistor at the sending end and an open circuit at the receiving end. So, by (7), $-1 < r_s < 1$ and $r_r = 1$. Each forward wave is totally reflected at the receiving end with no change of sign. If $R_s \neq Z_0$, each backward wave is reflected and diminished by the factor r_s (with a change of sign if $R_s < Z_0$) at the sending end. There is no reflection if $R_s = Z_0$ so that $r_s = 0$.

Altogether then, if $\mathbf{t} = [t_n]$ where $t_n = O(n)$, the voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ at the jth sample point is equal to a finite sum

$$2(1 + r_s + r_s^2 + r_s^3 + \cdots + r_s^p).$$

On the other hand, if $t_n/n \to \infty$ as $n \to \infty$, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is infinitesimally close to $2/(1 - r_s)$.

Example 6.2. Again, let $w(t) = 1_+(t)$, but this time set $R_s = R_r = 0$; that is, R_s and R_r are shorts. Thus, $r_s = r_r = -1$. So, $r_s^m r_r^m = 1$ and $r_s^m r_r^{m+1} = -1$ in (8).

In fact, for any $\mathbf{t} = [t_n]$ with $t_n = O(n)$ and for any sample point $\mathbf{x}_j$, only finitely many reflected waves will have passed $\mathbf{x}_j$. Consequently, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ equals either $+1$ or 0 depending upon whether the last reflected wave occurring at $\mathbf{x}_j$ at time $\mathbf{t}$ is either a forward wave or a backward wave respectively.

On the other hand, if $t_n/n \to \infty$ as $n \to \infty$, then the hyperreal $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ has as a representative sequence the infinite alternating sequence: $1, 0, 1, 0, 1, 0, \ldots$. This means that $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ equals either 1 or 0 depending upon the choice of the nonprincipal ultrafilter $\mathcal{F}$ used in the ultrapower construction of the hyperreals. So, now we have a two-way ambiguity as to the value of the hyperreal wave at any sample point $\mathbf{x}_j$ and time $\mathbf{t}$. Without specifying $\mathcal{F}$, the ultrapower construction cannot ascertain the hyperreal value of $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ between the two possible values 1 and 0.

7 Transfinite, Terminated, Distortionless or Lossless, Lines of Higher Ranks

Rather than considering a terminated ω^2-line, as we have in the preceding sections, let us now consider a terminated ω^μ-line, where μ is a natural number greater than 2. The analysis of such a line is much the same as before. Its only essential difference is in the more complicated expression for the terminations and truncations of the ω^μ-line.

Let us recall the recursive definition of an ω^μ-line, where again $\mu \in I\!N$, $\mu > 2$. By connecting ω-many ω^2-lines in cascade (with the infinite extremity of each ω^2-line connected to the input of the next ω^2-line), we obtain an ω^3-line. Similarly, for each $p = 4, \ldots, \mu$, an ω^p-line is obtained by connecting ω-many ω^{p-1}-lines in cascade again.

As before, we choose sample points with a uniform spacing Δx throughout the ω^μ-line. Then, a typical sample point has the index

$$j \;=\; \omega^{\mu-1}k_{\mu-1} \,+\, \omega^{\mu-2}k_{\mu-2} \,+\, \cdots \,+\, \omega k_1 \,+\, k_0 \tag{12}$$

where $k_{\mu-1}$ is the number of $\omega^{\mu-1}$-lines to the left of the $\omega^{\mu-1}$-line in which the sample point x_j appears, $k_{\mu-2}$ is the number of $\omega^{\mu-2}$-lines within the $\omega^{\mu-1}$-line in which the sample point x_j appears and to the left of the $\omega^{\mu-2}$-line in which the sample point x_j appears, and so on. In general, for $1 \leq \alpha \leq \mu - 1$, k_α is the number of ω^α-lines within the $\omega^{\alpha+1}$-line in which the sample point x_j appears and to the left of the ω^α-line in which the sample point x_j appears. Finally, k_0 is the number of sample points within the ω-line in which the sample point appears and to the left of that sample point.

Now, a terminated ω^μ-line is obtained by choosing any sample point of index

$$\omega^{\mu-1}l_{\mu-1} \,+\, \omega^{\mu-2}l_{\mu-2} \,+\, \cdots \,+\, \omega l_1 \,+\, l_0 \tag{13}$$

and deleting that part of the ω^μ-line beyond that point, which we now refer to as the receiving end of the terminated line. Also, we append a receiving-end resistor R_r, as shown in Figure 2. We now require that $l_{\mu-1} \geq 1$. Furthermore, we append a series connection of a voltage source $w(t)$, a sending end resistor R_s, and a switch to be closed at $t = 0$, again as shown in Figure 2. The terminated line is taken to be initially at rest, as before.

The index (12) of any sample point in the terminated line satisfies the following restriction: $0 \leq k_p \leq l_p$ for $p = 0, 1, \ldots, \mu - 1$.

The next step is to choose a truncation of the terminated ω^μ-line. A simple way of doing this is to replace the "ω-many" phrase in the definition of an ω^μ-line by the phrase "n-many," where $n \in \mathbb{N}$. In this case, each ω^p-line ($p = 1, \ldots, \mu - 1$) is replaced by an n^p-line and the output of each n^p-line is connected to the input of the next n^p-line—if there is a next n^p-line. Thus, the length L_n of the terminated ω^μ-line is

$$L_n \;=\; (n^{\mu-1}l_{\mu-1} \,+\, n^{\mu-2}l_{\mu-2} \,+\, \ldots \,+\, nl_1 \,+\, l_0)\,\Delta x. \tag{14}$$

The distance from the sending end to any sample point in the terminated line is

$$K_n \;=\; (n^{\mu-1}k_{\mu-1} \,+\, n^{\mu-2}k_{\mu-2} \,+\, \ldots \,+\, nk_1 \,+\, k_0)\,\Delta x. \tag{15}$$

where $0 \leq k_p \leq l_p$ ($p = 0, 1, \ldots, \mu - 1$), as before. Thus, the distance from that sample point to the receiving end is $L_n - K_n$.

The analysis of the terminated ω^μ-line now proceeds exactly as that for an ω^2-line except that $L_n = (l_1 n + l_0)\Delta x$ is replaced by (14) and $(k_1 n + k_0)\Delta x$ is replaced by (15). The conclusions for the distortionless line (resp. lossless line) are the same as those given in Sec. 5 (resp Sec. 6).

The next stage is in this examination of terminated transfinite lines is the examination of a terminated ω^ω-line. (See [2, page480] for the definition of an ω^ω-line.) However, such a termination yields simply a terminated ω^μ-line for some natural number μ, and nothing new transpires.

On the other hand, something more arises for an $\omega^{\mu+1}$-line. This is a cascade of ω-many ω^ω-lines. By terminating such a line at any one of its sample points, we will have l_ω-many ω^ω-lines to the left of that sample point, followed by an ω^μ-line up to the sample point. We can truncate this $\omega^{\mu+1}$-line, too, following the procedure stated in [2, page 481] to get a finite line that expands to fill out the $\omega^{\mu+1}$-line as the truncation index n tends to ∞. Next, we can then consider an $\omega^{\omega+2}$-line, ... , an ω^{ω^2}-line, and so on. We will not pursue these extensions, because the conclusions in every case are the same as those obtained heretofore.

8 The General Case of a Transfinite, Terminated, Transmission Line

Let us now comment very briefly about the complications that arise when the terminated transfinite line is neither distortionless nor lossless. Equation (8) still holds, but now

$$Z_0 \;=\; \sqrt{\frac{ls + r}{cs + g}}$$

and

$$\gamma \;=\; \sqrt{(ls + r)(cs + g)}$$

are irrational functions when l, c, and r are positive and g is nonnegative. So, too, are the reflection coefficients r_s and r_r irrational when $0 < R_s < \infty$ and $0 < R_r < \infty$. Thus, the terms in (8) cannot be identified as a traveling wave as given by Equations (3), (4), and

(5) in [2] when $w(t) = 1_+(t)$. What can be concluded in this general case remains an open question presently.

References

[1] E. Weber, *Linear Transient Analysis*, Vol. II, John Wiley: New York, 1956.

[2] A.H. Zemanian, Hyperreal transients on transfinite distributed transmission lines and cables, *International Journal of Circuit Theory and Applications*, **31** (2003), 473-482.

Figure Captions

Figure 1. An ω^2-line. It consists of infinitely many ω-lines (i.e., one-way infinite transmission lines) connected in cascade. The small circles indicate connections between the infinite extremities of the ω-lines with the inputs of the following ω-lines.

Figure 2. A terminated transfinite line; in particular, an $(\omega l_1 + l_0)$-line. The number of ω-lines is $l_1 \geq 1$, and the length of the final finite line is $l_0 \Delta x$, where Δx is the distance between sample points.

Figure 3. A finite transmission line terminated at its sending end ($x = 0$) by a voltage source $w(t)$ in series with a nonnegative resistor R_s and a switch and terminated at its receiving end ($x = L$) by a nonnegative resistor R_r (possibly $R_r = \infty$.)

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